Problem Set 2 due September 16, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: (a) Compute the inverses of the matrices:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix}$$

for various numbers a, b, c, a', b', c'. **You must use the Gauss-Jordan elimination procedure** outlined on page 11 of the Lecture notes (or page 86 of the textbook), that is, by starting from the augmented matrices $\begin{bmatrix} L & I \end{bmatrix}$ and $\begin{bmatrix} U & I \end{bmatrix}$. *(10 points)*

Solution: We perform the Gauss-Jordan elimination procedure for L:

$$\begin{bmatrix} L & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ c & b & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - ar_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ c & b & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - cr_1 - br_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & ab - c & -b & 1 \end{bmatrix},$$

so

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab - c & -b & 1 \end{bmatrix}.$$

Similarly, for U:

$$\begin{bmatrix} U & I \end{bmatrix} = \begin{bmatrix} 1 & a' & c' & 1 & 0 & 0 \\ 0 & 1 & b' & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - b'r_3} \begin{bmatrix} 1 & a' & c' & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -b' \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - a'r_2 - c'r_3} \begin{bmatrix} 1 & 0 & 0 & 1 & -a' & a'b' - c' \\ 0 & 1 & 0 & 0 & 1 & -b' \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

so

$$U^{-1} = \begin{bmatrix} 1 & -a' & a'b' - c' \\ 0 & 1 & -b' \\ 0 & 0 & 1 \end{bmatrix}.$$

Grading Rubric: 5 points for each matrix (5/5 for doing the procedure correctly, 2/5 if somewhat correct, 0/5 if completely incorrect attempt at the procedure).

(b) Compute the inverse of the matrix:

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

for any three non-zero numbers d_1, d_2, d_3. *(5 points)*

Solution:

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}.$$

Grading Rubric: 5/5 for exhibiting the correct inverse, 0/5 if incorrect.

(c) Consider the 3×3 matrix $A = LDU$, where L, D and U are as above. Write A^{-1} as a product of three matrices, and as a single matrix. Note that the formula for A^{-1} as a single matrix will not be particularly pretty, but it will give you some practice with matrix multiplication. *(10 points)*

Solution: In general, for invertible matrices M_1, M_2, their product M_1M_2 is invertible and $(M_1M_2)^{-1} = M_2^{-1}M_1^{-1}$. Hence

$$A^{-1} = (LDU)^{-1} = U^{-1}D^{-1}L^{-1}.$$

As a single matrix, this is:

$$\begin{bmatrix} 1 & -a' & a'b'-c' \\ 0 & 1 & -b' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{d_1} & -\frac{a'}{d_2} & \frac{a'b'-c'}{d_3} \\ 0 & \frac{1}{d_2} & -\frac{b'}{d_3} \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{d_1} + \frac{aa'}{d_2} + \frac{(a'b'-c')(ab-c)}{d_3} & -\frac{a'}{d_2} - \frac{b(a'b'-c')}{d_3} & \frac{a'b'-c'}{d_3} \\ -\frac{a}{d_2} - \frac{b'(ab-c)}{d_3} & \frac{1}{d_2} + \frac{bb'}{d_3} & -\frac{b'}{d_3} \\ \frac{ab-c}{d_3} & \frac{-b}{d_3} & \frac{1}{d_3} \end{bmatrix}.$$

Grading Rubric: 5 points for the product of three matrices (5/5 for the correct answer, 0/5 if incorrect), 5 points for the single matrix (5/5 if completely correct, 3/5 if mostly correct, 0/5 if mostly incorrect).

Problem 2: (a) If:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \tag{1}$$

and $XX' = I$, then what is the formula for a', b', c', d' in terms of a, b, c, d? You can look this up. *(5 points)*

Solution: We have that $a' = \frac{d}{ad-bc}$, $b' = -\frac{b}{ad-bc}$, $c' = -\frac{c}{ad-bc}$, and $d' = \frac{a}{ad-bc}$.

Grading Rubric: 5/5 for correct expressions, 0/5 if (even partially) incorrect.

(b) Now prove the aforementioned formula, by directly showing that the products of the two matrices (XX' and $X'X$) in (1) are both the unit matrix I (when computing the product, you must replace a', b', c', d' by their aforementioned formulas in terms of a, b, c, d). *(5 points)*

Solution: We have that

$$XX' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} a\frac{d}{ad-bc} - c\frac{b}{ad-bc} & -a\frac{b}{ad-bc} + b\frac{a}{ad-bc} \\ c\frac{d}{ad-bc} - d\frac{c}{ad-bc} & -c\frac{b}{ad-bc} + d\frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$X'X = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\frac{d}{ad-bc} - c\frac{b}{ad-bc} & b\frac{d}{ad-bc} d\frac{b}{ad-bc} \\ -a\frac{c}{ad-bc} + c\frac{a}{ad-bc} & -b\frac{c}{ad-bc} + d\frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Grading Rubric: 5/5 for computing both products correctly, 0/5 if either one is done incorrectly.

(c) Now consider the following block matrix:

$$X = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

where A and B are invertible, square matrices, C is a (not necessarily square) matrix, and 0 is a (also not necessarily square) zero matrix. Find (and prove) a formula for X^{-1} in terms of A, B, C and their inverses. What does your formula say in the explicit case:

$$X = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 5 \end{bmatrix}$$

(make sure you say what A, B, C are in this case)? *(10 points)*

Solution: We have that

$$X^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix}.$$

Indeed,

$$XX^{-1} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1} & -AA^{-1}CB^{-1} + CB^{-1} \\ 0 & BB^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$X^{-1}X = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A^{-1}A & A^{-1}C - A^{-1}CB^{-1}B \\ 0 & B^{-1}B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

so $XX^{-1} = XX^{-1} = I$.

In the explicit case, we may let $A = \begin{bmatrix} 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$. Then, because $A^{-1} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$ and from the formula in (b) $B^{-1} = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$, we have that

$$A^{-1}CB^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -\frac{5}{2} \end{bmatrix}.$$

Thus

$$X^{-1} = \begin{bmatrix} \frac{1}{2} & 6 & \frac{5}{2} \\ 0 & -5 & -2 \\ 0 & -3 & -1 \end{bmatrix}.$$

Grading Rubric: 5 points for identifying A, B, C (5/5 if correct, 0/5 if any are incorrect), 5 points for the inverse (5/5 if completely correct, 3/5 if partially correct).

Problem 3: (a) Instead of doing row operations, one can do column operations on a matrix. For example, start with:

$$A = \begin{bmatrix} \boxed{1} & 2 & 0 \\ -1 & \boxed{\boxed{-1}} & 3 \\ 2 & -3 & -5 \end{bmatrix}$$

and add appropriate multiples of the first column to the second and third columns, in such a way that all entries to the right of the box vanish. Then add an appropriate multiple of the second column to the third column, so that all entries to the right of the double box vanish. Carry out this process by showing all the steps. *(10 points)*

Solution:

$$\begin{bmatrix} \boxed{1} & 2 & 0 \\ -1 & \boxed{\boxed{-1}} & 3 \\ 2 & -3 & -5 \end{bmatrix} \xrightarrow{c_2 - 2c_1} \begin{bmatrix} \boxed{1} & 0 & 0 \\ -1 & \boxed{\boxed{1}} & 3 \\ 2 & -7 & -5 \end{bmatrix} \xrightarrow{c_3 - 3c_2} \begin{bmatrix} \boxed{1} & 0 & 0 \\ -1 & \boxed{\boxed{1}} & 0 \\ 2 & -7 & 16 \end{bmatrix}.$$

Grading Rubric: 5 points for each step (5/5 if correctly done, 0/5 if incorrect).

(b) Describe each of the steps in the process above as multiplying A on the right with an appropriate matrix. *(10 points)*

Solution: The first step corresponds to multiplication by

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on the right, and the second corresponds to multiplication by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Grading Rubric: 5 points for each matrix (5/5 if correct, 2/5 if the key entry is off by a sign, 0/5 if completely incorrect).

Problem 4: (a) Consider the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

and write it as the sum of a symmetric matrix and an anti-symmetric matrix (recall that while a symmetric matrix is one such that $S = S^T$, an anti-symmetric matrix is one such that $A = -A^T$). *(10 points)*

Solution: Let

$$S = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & 7 \\ 5 & 7 & 9 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Grading Rubric: 5 points for each matrix (5/5 if correct, 0/5 if incorrect).

(b) For a general matrix X, suppose you want to write it as $X = S + A$, where S is symmetric and A is anti-symmetric. Can you find formulas for S and A in terms of X only? *(5 points)*

Solution: Let $S = \frac{X+X^T}{2}$ and $A = \frac{X-X^T}{2}$. Then, clearly $S + A = X$, and $S^T = \frac{X^T+(X^T)^T}{2} = \frac{X^T+X}{2} = S$ and $A^T = \frac{X^T-(X^T)^T}{2} = \frac{X^T-X}{2} = -A$.

Grading Rubric: 5/5 if both expressions are correct, 0/5 if either one is incorrect.

Problem 5: (a) Consider the matrix:

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix}$$

and compute its LU factorization. *(10 points)*

Solution: We first perform Gauss-Jordan elimination on the matrix:

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} \xrightarrow{r_3 - 2r_2} \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} \xrightarrow{r_4 - 3r_3} \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

which expressed as matrix operations corresponds to the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

so

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

or

$$\begin{bmatrix}1&5&0&0\\1&7&6&0\\0&4&15&7\\0&0&9&25\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix}.$$

Thus, since

$$\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&2&1&0\\0&0&3&1\end{bmatrix},$$

letting

$$L=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&2&1&0\\0&0&3&1\end{bmatrix}\qquad U=\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix}$$

we find that LU equals the given matrix.

Grading Rubric: 10/10 if completely correct, 5/10 for the correct Gauss-Jordan elimination (but incorrect matrices), 0/10 if mostly incorrect attempt at computing the factorization.

(b) Based on the example in part (a), how do you think the LU factorization of a matrix of the form:

$$\begin{bmatrix}b_1&c_1&0&0&0&0&0\\a_2&b_2&c_2&0&0&0&0\\0&a_3&b_3&c_3&0&0&0\\0&0&a_4&b_4&c_4&0&0\\0&0&0&a_5&b_5&c_5&0\\0&0&0&0&a_6&b_6&c_6\\0&0&0&0&0&a_7&b_7\end{bmatrix}$$

will look like? (Just a general guess on how the matrices L and U will look will suffice for now. Hint: L and U will have a lot of zeroes. Where do you think they are located?) *(5 points)*

Solution: The L matrix will have 1s along the main diagonal, possibly nonzero entries on the diagonal below it, and zeros for the remaining entries. The U matrix will have the entries $b_1, \ldots, b_7$ along the main diagonal as in the original matrix, and $c_1, \ldots, c_6$ along the diagonal immediately above it, as in the original matrix.

Grading Rubric: 5/5 for mentioning somehow that the L matrix will consist of zeros except possibly on the main diagonal and diagonal right below it and that the U will consist of zeros except possibly on the main diagonal and diagonal right above it.

(c) In the generality of part (b), work out recursive formulas for the coefficients of L and U in terms of $a_2, ..., a_7$, $b_1, ..., b_7$, $c_1, ..., c_6$. *(5 points)*

Solution: Letting

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} y_1 & z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_2 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_3 & z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4 & z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_5 & z_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_6 & z_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & y_7 \end{bmatrix},$$

explicit computation shows that $z_j = c_j$ for all j, $y_1 = b_1$, $x_j = \frac{a_{j+1}}{y_j}$ for all j, and $y_k = b_k - x_{k-1}z_{k-1}$ for $k > 1$.

Grading Rubric: 5/5 if all entries are correct, 3/5 if most are correct, 0/5 if most are incorrect.